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The Casimir energy for a rectangular cavity at finite temperature

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Abstract

In this paper the Casimir effect arising from the case of Dirichlet boundary conditions confining the massless scalar field at finite temperature is reexamined for a $(D-1)$-dimensional rectangular cavity with equal or unequal finite p edges and different spacetime dimensions D. We derive an expression for the Casimir energy for a p-dimensional cavity at nonzero temperature. We show that the sign of the Casimir energy remains positive irrespective of whether p is odd or even if the thermal corrections to the standard Casimir effect are sufficiently large. Furthermore, we also find the temperature influences on choosing edges which lead to the Casimir energy being positive or negative.

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1. Introduction

The Casimir effect is a fundamental aspect of quantum field theory in confined geometries. Historically, the effect of boundaries was investigated by Casimir [1]. The vacuum fluctuations as embodied in the Casimir effect have also been a subject of extensive research [2]. It has been showed that the Casimir effect is essentially a polarization of the vacuum of quantized fields which arises because of a change in the spectrum.

The Casimir effect as a strong function of geometry has been researched both theoretically and experimentally for a long time [4, 5] and for a rectangular cavity. The sign of the Casimir energy depends on the spacetime dimensions and configuration of boundaries that confine the field. When a massless scalar field is quantized inside a box with p edges of equal length $L_1 = L_2 = \cdots = L_p = L$ in a D-dimensional spacetime and the characteristic length of $D-1-p$ edges satisfies $\lambda \gg L$, the sign of the Casimir energy E_p^D is always negative [3]. It is very interesting that for Dirichlet boundary conditions the sign of E_p^D depends on whether p is even or odd [4]. In an earlier paper we discussed the Casimir effect of a massless scalar field for a rectangular cavity with unequal edges and Dirichlet boundary conditions in a D-dimensional

Minkowski spacetime [5]. We showed that the sign of the Casimir energy depends on ratios of edges for the p-dimensional cavity with unequal edges instead of only depending on whether p is odd or even.

Quantum field theory at finite temperature shares many of these effects. It is necessary to discuss the Casimir energy under a nonzero temperature environment [3]. Recently, the experimental observation of the Casimir effect was greatly improved [5, 8, 9]. Thermal corrections to the standard Casimir effect cannot be neglected in many cases [3, 10–13]. In this paper we discuss an important issue of the Casimir effect at finite temperature in detail. We find that there are more and greater differences between the results in the zero-temperature case and the nonzero ones.

The description of the Casimir effect for a rectangular cavity with equal or unequal edges will change at different temperatures. Here we consider the thermal corrections to the Casimir effect for a rectangular cavity by means of finite-temperature field theories [6, 7, 13]. By regularizing the Casimir energy density, we find that the thermal influence cannot be omitted for sufficiently high temperatures. First we derive the Casimir energy for a p-dimensional cavity at finite temperature in D-dimensional spacetime. Secondly we discuss the $p = 2, 3$ cases carefully. Finally the conclusions are listed.

2. The Casimir energy at finite temperature in a D-dimensional Minkowski spacetime

It is convenient to describe the scalar fields in thermal equilibrium by making use of the imaginary time formalism. First we introduce a partition function for a system [3,6,7,13]

$$Z = N \int_{\text{periodic}} D\phi \exp\left[\int_0^\beta \mathrm{d}\tau \int \mathrm{d}^3x\, L(\phi, \partial_E\phi)\right] \tag{1}$$

where L is the Lagrangian density for the system under consideration, N a constant and 'periodic' means,

$$\phi(0, \vec{x}) = \phi(\tau, \vec{x}) \tag{2}$$

where $\beta = \frac{1}{T}$ is the inverse of the temperature. The system, a Hermitian massless scalar field, can be described through the massless Klein–Gordon equation [4, 5]

$$(\partial_t^2 - \partial_i^2)\phi(t, x^a, x^T) = 0 \tag{3}$$

where $i = 1, 2, \ldots, D-1$; $a = 1, 2, \ldots, p$; $T = p+1, \ldots, D-1$. The field satisfying the Dirichlet boundary conditions $\phi(t, x^a, x^T)|_{\partial\Omega} = 0$ is confined to the interior of a $(D-1)$-dimensional rectangular cavity Ω with p edges of finite lengths $L_1, L_2, \ldots, L_p$ and $D-1-p$ edges with characteristic lengths of order $\lambda \gg L_a$. The modes of the massless scalar field,

$$\phi_{\{n\}} = \sin\frac{n_1\pi x_1}{L_1} \sin\frac{n_2\pi x_2}{L_2} \cdots \sin\frac{n_p\pi x_p}{L_p} \mathrm{e}^{\mathrm{i}k_T\cdot x_T}\mathrm{e}^{-\mathrm{i}\omega_k t} \tag{4}$$

$$\omega_k^2 = k_T^2 + \left(\frac{n_1\pi}{L_1}\right)^2 + \left(\frac{n_2\pi}{L_2}\right)^2 + \cdots + \left(\frac{n_p\pi}{L_p}\right)^2 + \left(\frac{2n\pi}{\beta}\right)^2 \tag{5}$$

where $\{n\}$ denotes short-hand notation for $n_1, n_2, \ldots, n_p$, and n_a is a positive integer. The generalized zeta function can be written as

$$\zeta(s; -\partial_E) = \mathrm{Tr}(-\partial_E)^{-s} \tag{6}$$

where

$$\partial_E = \frac{\partial^2}{\partial\tau^2} + \nabla^2 \tag{7}$$

and

$$\tau = \mathrm{i}t. \tag{8}$$

According to the solutions to the equations of motion and equation (6), the generalized zeta function reads

$$\zeta(s;-\partial_E)=\int \mathrm{d}^{D-p-1}k\sum_{\{n\}=1}^{\infty}\sum_{n=-\infty}^{\infty}\left[k_T^2+\left(\frac{n_1\pi}{L_1}\right)^2+\left(\frac{n_2\pi}{L_2}\right)^2+\cdots+\left(\frac{n_p\pi}{L_p}\right)^2+\left(\frac{2n\pi}{\beta}\right)^2\right]^{-s}. \tag{9}$$

Following [5, 13], the function can be expressed in terms of Epstein zeta functions,

$$\zeta(s;-\partial_E)=\frac{\pi^{\frac{D-p-1}{2}}\Gamma\left(s-\frac{D-p-1}{2}\right)}{\Gamma(s)}E_p\left(\frac{2s-D+p+1}{2};\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2},\ldots,\frac{\pi^2}{L_p^2}\right)+\frac{2\pi^{\frac{D-p-1}{2}}\Gamma\left(s-\frac{D-p-1}{2}\right)}{\Gamma(s)}\times E_{p+1}\left(\frac{2s-D+p+1}{2};\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2},\ldots,\frac{\pi^2}{L_p^2},\frac{4\pi^2}{\beta^2}\right) \tag{10}$$

where the Epstein zeta function is defined as

$$E_N(s;a_1,a_2,\ldots,a_N)=\sum_{n_1,n_2,\ldots,n_N=1}^{\infty}\left(\sum_{j=1}^{N}a_jn_j^2\right)^{-s} \tag{11}$$

the total energy density of the system with thermal corrections is

$$\begin{aligned}\varepsilon_p^D=&-\frac{\partial}{\partial\beta}\left(\left.\frac{\partial\zeta(s;-\partial_E)}{\partial s}\right|_{s=0}\right)=-\frac{\pi^{\frac{D-p-2}{2}}}{2}\Gamma\left(\frac{p-D}{2}\right)E_p\left(\frac{p-D}{2};\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2},\ldots,\frac{\pi^2}{L_p^2}\right)\\&+2^{\frac{D-p+1}{2}}\pi^{\frac{D-p-1}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\frac{D-p-1+2k}{2}\beta^{-\frac{D-p+1+2k}{2}}\\&\times\prod_{j=1}^{k}[(p-D)^2-(2j-1)^2]\sum_{m,n_1,n_2,\ldots,n_p=1}^{\infty}m^{\frac{p-D-2k-1}{2}}\\&\times\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\cdots+\frac{\pi^2}{L_p^2}n_p^2\right)^{\frac{D-p-1-2k}{4}}\\&\times\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\cdots+\frac{\pi^2}{L_p^2}n_p^2\right)^{\frac{1}{2}}\right]\\&+2^{\frac{D-p+1}{2}}\pi^{\frac{D-p-1}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\left(\frac{1}{\beta}\right)^{\frac{D-p-1+2k}{2}}\prod_{j=1}^{k}[(p-D)^2-(2j-1)^2]\\&\times\sum_{m,n_1,n_2,\ldots,n_p=1}^{\infty}m^{\frac{p-D-2k+1}{2}}\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\cdots+\frac{\pi^2}{L_p^2}n_p^2\right)^{\frac{D-p+1-2k}{4}}\\&\times\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\cdots+\frac{\pi^2}{L_p^2}n_p^2\right)^{\frac{1}{2}}\right].\end{aligned} \tag{12}$$

The last two terms are thermal corrections. Let $T\to 0$ or $\beta\to\infty$, then equation (12) becomes the same as that of [5].

3. The sign of the Casimir energy density of a $p = 2$ cavity at finite temperature

First we consider the $p = 2$ case in which the field is confined to the interior of a $(D-1)$-dimensional rectangular cavity with two edges of finite lengths L_1, L_2 and $D-3$ edges with characteristic lengths of order $\lambda \gg L_1, L_2$. The Casimir energies for the fields confined in a rectangular cavity with equal or unequal edges at zero temperature have been obtained [4, 5]. Here we study the influence of the thermal corrections on the Casimir effect. According to equation (12) the same case at finite temperature can be described as

$$\varepsilon^D_{p=2} = -\frac{1}{2}\pi^{\frac{D-4}{2}}\Gamma\left(\frac{2-D}{2}\right)E_2\left(\frac{2-D}{2};\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2}\right) + \Sigma_1(2,D,\beta) + \Sigma_2(2,D,\beta) \tag{13}$$

where

$$\Sigma_1(2,D,\beta) = 2^{\frac{D-1}{2}}\pi^{\frac{D-3}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\frac{D+2k-3}{2}\left(\frac{1}{\beta}\right)^{\frac{D+2k-1}{2}}\prod_{j=1}^{k}[(2-D)^2-(2j-1)^2]$$
$$\times\sum_{m,n_1,n_2=1}^{\infty}m^{-\frac{D+2k-1}{2}}\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2\right)^{\frac{D-2k-3}{4}}\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2\right)^{\frac{1}{2}}\right] \tag{14}$$

$$\Sigma_2(2,D,\beta) = 2^{\frac{D-1}{2}}\pi^{\frac{D-3}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\left(\frac{1}{\beta}\right)^{\frac{D+2k-3}{2}}\prod_{j=1}^{k}[(2-D)^2-(2j-1)^2]$$
$$\times\sum_{m,n_1,n_2=1}^{\infty}m^{-\frac{D+2k-3}{2}}\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2\right)^{\frac{D-2k-1}{4}}\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2\right)^{\frac{1}{2}}\right]. \tag{15}$$

When the temperature is equal to zero, it was shown that the energy density is positive for $D \leqslant 6$ and becomes negative for integer values $D \geqslant 7$ in the case of an equal-edge cavity [4]. Let us start to discuss the topic at nonzero temperature. For the $L_1 = L_2 = L$ case, having introduced a dimensionless variable $\xi = \frac{L}{\pi\beta}$ named the scaled temperature and regularized the expression, we obtain the Casimir energy

$$\varepsilon^D_{p=2} = \frac{1}{4}\pi^{\frac{D-5}{2}}\left(\frac{1}{L}\right)^{D-2}\Gamma\left(\frac{D-1}{2}\right)\zeta(D-1) - \frac{1}{4}\pi^{\frac{D-6}{2}}\left(\frac{1}{L}\right)^{D-2}\Gamma\left(\frac{D}{2}\right)\zeta(D)$$
$$-\pi^{D-3}\left(\frac{1}{L}\right)^{D-2}\sum_{n_1,n_2=1}^{\infty}\left(\frac{n_2}{n_1}\right)^{\frac{D-1}{2}}K_{\frac{D-1}{2}}(2\pi n_1 n_2) + \Sigma_1(2,D,\xi) + \Sigma_2(2,D,\xi) \tag{16}$$

where

$$\Sigma_1(2,D,\xi) = 2^{\frac{D-1}{2}}\frac{\pi^{\frac{3D-7}{2}}}{L^{D-2}}\xi^{\frac{D-1}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\frac{D+2k-3}{2}\xi^k\prod_{j=1}^{k}[(2-D)^2-(2j-1)^2]$$
$$\times\sum_{m,n_1,n_2=1}^{\infty}m^{\frac{1-D-2k}{2}}(n_1^2+n_2^2)^{\frac{D-2k-3}{4}}\exp\left[-\frac{m}{\xi}(n_1^2+n_2^2)^{\frac{1}{2}}\right] \tag{17}$$

$$\Sigma_2(2,D,\xi) = 2^{\frac{D-1}{2}}\frac{\pi^{\frac{3D-7}{2}}}{L^{D-2}}\xi^{\frac{D-3}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\xi^k\prod_{j=1}^{k}[(2-D)^2-(2j-1)^2]$$
$$\times\sum_{m,n_1,n_2=1}^{\infty}m^{\frac{3-D-2k}{2}}(n_1^2+n_2^2)^{\frac{D-2k-1}{4}}\exp\left[-\frac{m}{\xi}(n_1^2+n_2^2)^{\frac{1}{2}}\right]. \tag{18}$$

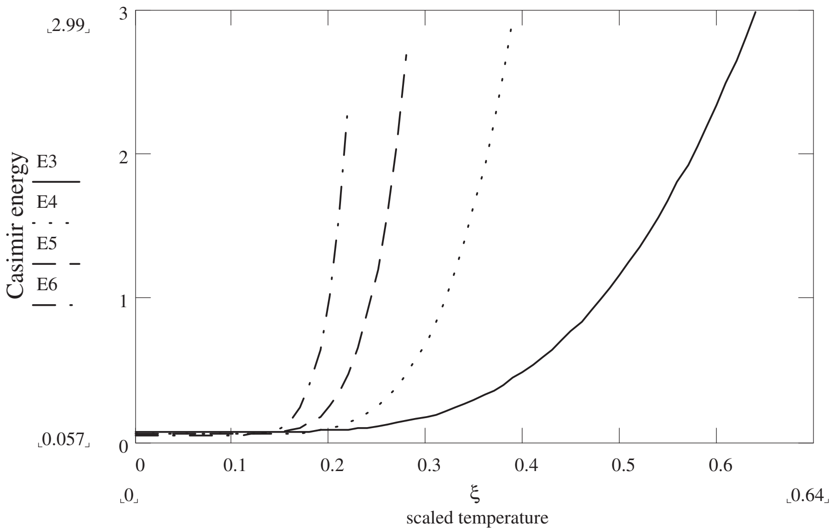

Figure 1. The solid, dot, dashed and dot–dashed curves of the Casimir energy density as functions of scaled temperature $\xi = \frac{L}{\pi\beta}$ in D-dimensional spacetime for $D = 3, 4, 5, 6$ respectively.

The results of the converging series can be calculated efficiently with the help of the Macdonald's function expression

$$K_\nu(z) = \sqrt{\frac{\pi}{2z}}\mathrm{e}^{-z}\sum_{k=0}^{\infty}\frac{1}{k!}\frac{1}{(8z)^k}\prod_{j=1}^{k}[4\nu^2 - (2j-1)^2]. \tag{19}$$

The terms denoted as $\Sigma_1(2,\ D,\ \xi)$ and $\Sigma_2(2,\ D,\ \xi)$ respectively, converge very quickly owing to the rapid exponential convergence and only the first several summands need to be taken into account according to the numerical calculation. The numerical calculations for the energy density lead to the data presented in figures 1 and 2 which show the Casimir energy as an increasing function of the scaled temperature for $p = 2$ and $D = 3, 4, \ldots, 9$, where we choose L as the unit length. The special scaled temperatures ξ_0 for spacetimes with different dimensions can also be calculated from equation (16). In the cases of $D \geqslant 7$, the Casimir energy will become positive if the scaled temperatures are chosen as $\xi > \xi_0$, and now the values of ξ_0 for spacetimes with $D = 3, 4, \ldots, 9$ are listed in table 1. It is important that there exists no particular critical value of spacetime dimension D_c in the case of equal edges if the temperature is large enough.

Secondly we consider the cavity with the unequal edges case at finite temperature. We introduce another dimensionless variable $\mu_D = \frac{L_2}{L_1}$ and set L_1 to unity. Having regularized and solved equations (13)–(15), by means of the Mellin transformation, we obtain the relations between μ_D^0 and temperature, with μ_D^0 growing as scaled temperature increases for some spacetime as shown in figure 3. In a D-dimensional spacetime at finite temperature, the Casimir energy density will be negative if $\frac{L_2}{L_1} > \mu_D^0$, and if $\frac{L_2}{L_1} < \mu_D^0$, the energy density will remain positive.

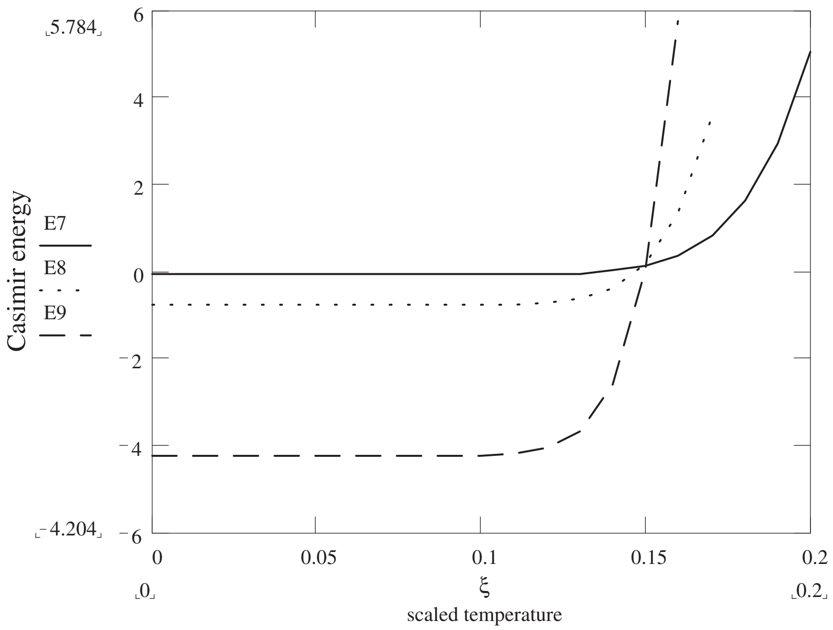

Figure 2. The solid, dot and dashed curves of the Casimir energy density as functions of scaled temperature $\xi = \frac{L}{\pi\beta}$ in a D-dimensional spacetime for $D = 7, 8, 9$ respectively.

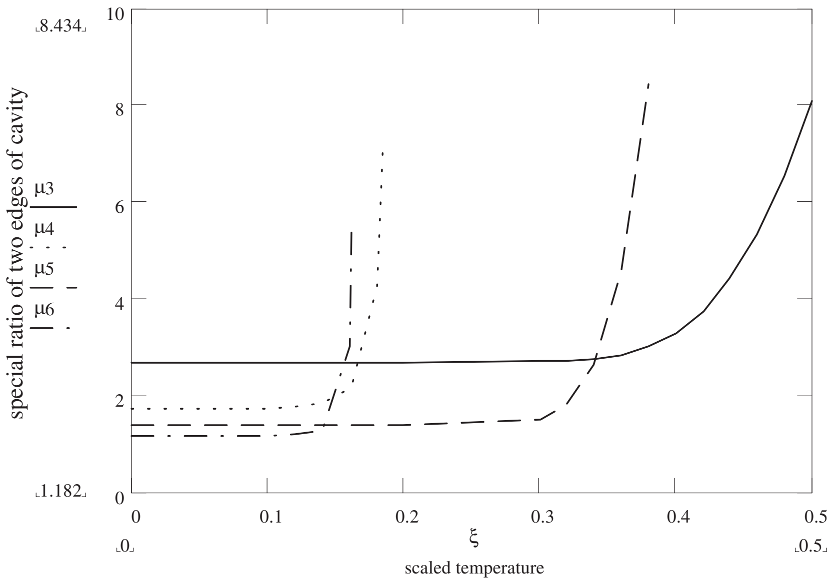

Figure 3. The solid, dot, dashed and dot–dashed curves of the special ratio of two edges of a $p = 2$ cavity μ_D^0 as a function of scaled temperature $\xi = \frac{L}{\pi\beta}$ in D-dimensional spacetime for $D = 3, 4, 5, 6$ respectively.

4. The sign of the Casimir energy density of a $p = 3$ cavity at finite temperature

When the number of finite and equal edges of a hypercube is odd and the temperature is equal to zero, it has been analytically shown that $\varepsilon_p^D < 0$ for any D [4]. Here we discuss the question of

Table 1. The special scaled temperatures ξ_0 for massless scalar fields satisfying Dirichlet boundary conditions inside a cavity with two equal edges at finite temperature in a D-dimensional spacetime.

D	ξ_0
7	0.136
8	0.148
9	0.15
10	0.15
11	0.15
12	0.15
13	0.15

Table 2. The special scaled temperatures ξ_0 for massless scalar fields satisfying Dirichlet boundary conditions inside a cavity with three equal edges at finite temperature in a D-dimensional spacetime.

D	ξ_0
4	0.277
5	0.213
6	0.184
7	0.169
8	0.161
9	0.157
10	0.155
11	0.154

whether the Casimir energy for odd-number edges with nonzero temperature gives rise to a positive or a negative sign. For simplicity, we take $p = 3$ in equation (12)

$$\begin{aligned}
\varepsilon^D_{p=3} = &-\frac{1}{2}\pi^{\frac{D-5}{2}}\Gamma\left(\frac{3-D}{2}\right)E_3\left(\frac{3-D}{2};\frac{\pi^2}{L_1^2},\frac{\pi^2}{L_2^2},\frac{\pi^2}{L_3^2}\right)\\
&+2^{\frac{D-2}{2}}\pi^{\frac{D-4}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\frac{D-4+2k}{2}\beta^{-\frac{D-2+2k}{2}}\prod_{j=1}^{k}[(3-D)^2-(2j-1)^2]\\
&\times\sum_{m,n_1,n_2,n_3=1}^{\infty}m^{\frac{2-D-2k}{2}}\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\frac{\pi^2}{L_3^2}n_3^2\right)^{\frac{D-4-2k}{4}}\\
&\times\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\frac{\pi^2}{L_3^2}n_3^2\right)^{\frac{1}{2}}\right]\\
&+2^{\frac{D-2}{2}}\pi^{\frac{D-4}{2}}\sum_{k=0}^{\infty}\frac{8^{-k}}{k!}\left(\frac{1}{\beta}\right)^{\frac{D-4+2k}{2}}\prod_{j=1}^{k}[(3-D)^2-(2j-1)^2]\\
&\times\sum_{m,n_1,n_2,n_3=1}^{\infty}m^{\frac{4-D-2k}{2}}\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\frac{\pi^2}{L_3^2}n_3^2\right)^{\frac{D-2k-2}{4}}\\
&\times\exp\left[-\beta m\left(\frac{\pi^2}{L_1^2}n_1^2+\frac{\pi^2}{L_2^2}n_2^2+\frac{\pi^2}{L_3^2}n_3^2\right)^{\frac{1}{2}}\right].
\end{aligned}\tag{20}$$

By means of the Mellin transform and substituting $\xi = \frac{L}{\pi\beta}$ and $L_1 = L_2 = L_3 = L$ into equation (20), we obtain the special scaled temperatures for different dimensions. The last two terms in equation (20) also converge quickly, similar to equation (16). Some special scaled temperatures ξ_0 are listed in table 2. The Casimir energy is positive for $\xi > \xi_0$ and negative for $\xi < \xi_0$. It has been shown that the Casimir energy can also become positive for the $p = 3$ odd-number-edged hypercube at a high enough temperature.

5. Conclusion

Here we have discussed the massless scalar field at finite temperature in a rectangular cavity. We derive the expression for the Casimir energy with thermal corrections for a p-dimensional cavity in a D-dimensional spacetime. Having dealt with the $p = 2, 3$ cases carefully, we show that the thermal corrections cannot be omitted if the temperature is large enough. For a $p = 2$ equal-edge cavity, although the dimension $D \geqslant 7$, the Casimir energy will become positive in a D-dimensional spacetime if the temperature is larger than a special value. We also show the relation between the ratio of two edges and temperature for an unequal-edge cavity. For the $p = 3$ equal-edge cavity, it is interesting that the Casimir energy can also change to positive if the temperature is sufficiently high.

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